

Infinitely many periodic solutions for second order Hamiltonian systems

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Abstract In this paper, we study the existence of infinitely many periodic solutions for second order Hamiltonian systems $\ddot{u} + \nabla_u V(t, u) = 0$, where $V(t, u)$ is either asymptotically quadratic or superquadratic as $|u| \rightarrow \infty$.

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1 Introduction and main results

We consider the following second order Hamiltonian systems

$$\begin{cases} \ddot{u} + \nabla_u V(t, u) = 0, & \forall t \in \mathbb{R}, \\ u(0) = u(T), \dot{u}(0) = \dot{u}(T), & T > 0, \end{cases} \quad (1.1)$$

where $V \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ is T -periodic and has the form

$$V(t, u) = \frac{1}{2} \langle U(t)u, u \rangle + W(t, u) \quad (1.2)$$

with $U(\cdot)$ a continuous T -periodic symmetric matrix. Here and in the sequel, $\langle \cdot, \cdot \rangle$ and $|\cdot|$ always denote the standard inner product and the associated norm in \mathbb{R}^N respectively.

In this paper, we will study the existence of infinitely many nontrivial solutions of (1.1) via the variant fountain theorems established in [27] under the assumption that $W(t, u)$ is even in u , i.e., $W(t, -u) = W(t, u)$ for all $(t, u) \in [0, T] \times \mathbb{R}^N$. We divide the problem into the following two cases.

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1.1 The asymptotically quadratic case

For the asymptotically quadratic case, we make the following assumptions:

(AQ₁) $W(t, u) \geq 0$ for all $(t, u) \in [0, T] \times \mathbb{R}^N$, and there exist constants $\mu \in (0, 2)$ and $R_1 > 0$ such that

$$\langle \nabla_u W(t, u), u \rangle \leq \mu W(t, u), \quad \forall t \in [0, T] \text{ and } |u| \geq R_1,$$

(AQ₂) $\lim_{|u| \rightarrow 0} \frac{W(t, u)}{|u|^2} = \infty$ uniformly for $t \in [0, T]$, and there exist constants $c_2, R_2 > 0$ such that

$$W(t, u) \leq c_2 |u|, \quad \forall t \in [0, T] \text{ and } |u| \leq R_2,$$

(AQ₃) $\liminf_{|u| \rightarrow \infty} \frac{W(t, u)}{|u|} \geq d > 0$ uniformly for $t \in [0, T]$.

We state our first main result as follows.

Theorem 1.1. *Assume that (AQ₁)–(AQ₃) hold and that $W(t, u)$ is even in u . Then (1.1) possesses infinitely many nontrivial solutions.*

1.2 The superquadratic case

For the superquadratic case, we assume

(SQ₁) There exist constants $a_1 > 0$ and $\nu > 2$ such that

$$|\nabla_u W(t, u)| \leq a_1(1 + |u|^{\nu-1}), \quad \forall t \in [0, T] \text{ and } u \in \mathbb{R}^N,$$

(SQ₂) $W(t, u) \geq 0$ for all $(t, u) \in [0, T] \times \mathbb{R}^N$, and $\lim_{|u| \rightarrow \infty} \frac{W(t, u)}{|u|^2} = \infty$ uniformly for $t \in [0, T]$,

(SQ₃) There exist constants $1 \leq \varrho \in (\nu - 2, \infty)$ and $b > 0$ such that

$$\liminf_{|u| \rightarrow \infty} \frac{\langle \nabla_u W(t, u), u \rangle - 2W(t, u)}{|u|^\varrho} \geq b \text{ uniformly for } t \in [0, T].$$

Our second main result reads as follows.

Theorem 1.2. *Suppose that (SQ₁)–(SQ₃) are satisfied and that $W(t, u)$ is even in u . Then (1.1) possesses infinitely many nontrivial solutions.*

With the aid of variational methods, the existence and multiplicity of periodic solutions for Hamiltonian systems have been extensively investigated in many papers (see [1–19, 21–23, 25, 26, 28] and the references therein).

For asymptotically quadratic case, under various twist conditions via Morse indices or Maslov-type indices, the authors obtained finitely many periodic solutions in [1, 13, 15, 28] without any evenness assumption, while in the presence of evenness, the authors in [26] studied the existence of infinitely many solutions for (1.1) under the conditions that $W(t, u)$ is sign-changing and in some sense of at most linear growth near infinity, which are totally different from our conditions (AQ_1) and (AQ_3) in Theorem 1.1. In [7], the authors also obtained infinitely many periodic solutions for first order Hamiltonian systems. We note that some conditions of Theorem 1.1 in [7] will not be satisfied when problem (1.1) with conditions (AQ_1) – (AQ_3) is transformed to the corresponding first order Hamiltonian system in [7].

For the superquadratic case, most of the results on the multiplicity of periodic solutions were obtained under the so-called Ambrosetti-Rabinowitz superquadratic condition near infinity with or without the evenness assumption (see e.g. [3–5, 7, 11, 12, 14, 16, 17, 19, 23]). As mentioned in [10], for first order Hamiltonian systems, the Ambrosetti-Rabinowitz superquadratic condition requires the Hamiltonian $H(t, z)$ to be superquadratic in all components of the variable $z = (p, q)$, which excludes the case for the second order Hamiltonian systems (1.1) with $H(t, p, q) = \frac{1}{2}|p|^2 + V(t, q)$. In [9], the author introduced a new superquadratic condition for first order Hamiltonian systems, which requires only a combined effect of Ambrosetti-Rabinowitz superquadratic nature in p and q with $z = (p, q)$ and can include the above case for second order Hamiltonian systems. Subsequently, under the superquadratic condition of this type, the authors in [10] obtained the existence of infinitely many periodic solutions with the evenness assumption. For second order Hamiltonian systems (1.1), we note that the Ambrosetti-Rabinowitz superquadratic condition is somewhat stronger than the superquadratic condition given by (SQ_2) and (SQ_3) in Theorem 1.2.

2 Variational setting and proofs of the main results

In this section, we will first recall some related preliminaries and establish the variational setting for our problem, and then give the proofs of the main results.

2.1 Preliminaries and Variational setting

Within this subsection, we will introduce the variational setting for problem (1.1). Recall that the space $H^1(S_T, \mathbb{R}^N)$ becomes a Hilbert space if it is equipped with the usual norm

$$\|u\|_1 = \left(\int_0^T (|\dot{u}|^2 + |u|^2) dt \right)^{1/2}, \quad \forall u \in H^1(S_T, \mathbb{R}^N),$$

where $S_T = \mathbb{R}/T\mathbb{Z}$.

Denote by \mathcal{A} the operator $-(d^2/dt^2) - U(t)$ on $L^2 \equiv L^2((0, T), \mathbb{R}^N)$ with domain $D(\mathcal{A}) = H^2(S_T, \mathbb{R}^N)$. It is known that \mathcal{A} is a selfadjoint operator with a sequence of eigenvalues (counted with multiplicity)

$$\lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty \quad (2.1)$$

and the corresponding system of eigenfunctions $\{e_n : n \in \mathbb{N}\} (\mathcal{A}e_n = \lambda_n e_n)$ forming an orthogonal basis in L^2 . Denote by $|\mathcal{A}|$ the absolute value of \mathcal{A} and let $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$ with domain $D(|\mathcal{A}|^{1/2})$. By the elliptic estimate and Theorem 3.6 in [24], we have

$$D(|\mathcal{A}|^{1/2}) = H^1(S_T, \mathbb{R}^N).$$

Furthermore, if we define on $H^1(S_T, \mathbb{R}^N)$ a new inner product and the associated norm by

$$\begin{aligned} (u, v)_0 &= (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u, v)_2, \\ \|u\|_0 &= (u, u)_0^{1/2}, \end{aligned}$$

then $\|\cdot\|_0$ is equivalent to the usual norm $\|\cdot\|_1$ on $H^1(S_T, \mathbb{R}^N)$, where $(\cdot, \cdot)_2$ denotes the usual inner product on L^2 . Set

$$n^- = \#\{i | \lambda_i < 0\}, \quad n^0 = \#\{i | \lambda_i = 0\}, \quad \bar{n} = n^- + n^0, \quad (2.2)$$

and let

$$L^2 = L^- \oplus L^0 \oplus L^+ \quad (2.3)$$

be the orthogonal decomposition in L^2 with

$$\begin{aligned} L^- &= \text{span}\{e_1, \dots, e_{n^-}\}, \quad L^0 = \text{span}\{e_{n^-+1}, \dots, e_{\bar{n}}\}, \\ L^+ &= (L^- \oplus L^0)^\perp = \overline{\text{span}\{e_{\bar{n}+1}, \dots\}}. \end{aligned}$$

Now we introduce on $H^1(S_T, \mathbb{R}^N)$ the following inner product and norm:

$$(u, v) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u^0, v^0)_2,$$

$$\|u\| = (u, u)^{1/2},$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ with respect to the decomposition (2.3). Let $E = H^1(S_T, \mathbb{R}^N)$, then E becomes a Hilbert space with the inner (\cdot, \cdot) . Clearly, norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent. Consequently, the norm $\|\cdot\|$ is also equivalent to the norm $\|\cdot\|_1$ on E . From now on $(E, (\cdot, \cdot), \|\cdot\|)$ becomes our working space.

Remark 2.1. It is easy to check that E possesses the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+ \quad (2.4)$$

with

$$E^- = L^-, \quad E^0 = L^0 \quad \text{and} \quad E^+ = E \cap L^+ = \overline{\text{span}\{e_{\bar{n}+1}, \dots\}} \quad (2.5)$$

where the closure is taken with respect to the norm $\|\cdot\|$. Evidently, the above decomposition is also orthogonal in L^2 .

By the Sobolev embedding theorem, we get directly the following lemma.

Lemma 2.2. *E is compactly embedded in $L^p \equiv L^p((0, T), \mathbb{R}^N)$ for $1 \leq p \leq \infty$ and hence there exists $\tau_p > 0$ such that*

$$|u|_p \leq \tau_p \|u\|, \quad \forall u \in E, \quad (2.6)$$

where $|\cdot|_p$ denotes the usual norm on L^p for all $1 \leq p \leq \infty$.

Now we define a functional Φ on E by

$$\begin{aligned} \Phi(u) &= \frac{1}{2} \int_0^T (|\dot{u}|^2 - \langle U(t)u, u \rangle) dt - \Psi(u) \\ &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \Psi(u) \quad \text{where } \Psi(u) = \int_0^T W(t, u) dt \end{aligned} \quad (2.7)$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$. Note that (AQ₁) and (AQ₃) imply

$$W(t, u) \leq c_1(1 + |u|^\mu), \quad \forall (t, u) \in [0, T] \times \mathbb{R}^N \quad (2.8)$$

for some $c_1 > 0$. Likewise, by (SQ₁), there exists a constant $a_2 > 0$ such that

$$W(t, u) \leq a_1(|u| + |u|^\nu) + a_2, \quad \forall (t, u) \in [0, T] \times \mathbb{R}^N. \quad (2.9)$$

In view of (2.8) (or (2.9)) and Lemma 2.2, Φ and Ψ are well defined. Furthermore, we have the following

Proposition 2.3. *Suppose that either (AQ₁) and (AQ₃) or (SQ₁) is satisfied. Then $\Psi \in C^1(E, \mathbb{R})$ and $\Psi' : E \rightarrow E^*$ is compact, and hence $\Phi \in C^1(E, \mathbb{R})$. Moreover,*

$$\Psi'(u)v = \int_0^T \langle \nabla_u W(t, u), v \rangle dt, \quad (2.10)$$

$$\begin{aligned} \Phi'(u)v &= (u^+, v^+) - (u^-, v^-) - \Psi'(u)v \\ &= (u^+, v^+) - (u^-, v^-) - \int_0^T \langle \nabla_u W(t, u), v \rangle dt \end{aligned} \quad (2.11)$$

for all $u, v \in E = E^- \oplus E^0 \oplus E^+$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ respectively, and critical points of Φ on E are solutions of (1.1).

Proof. The proof is standard and we refer to [6] and [20]. \square

Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\oplus_{j \in \mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. Consider the following C^1 -functional $\Phi_\lambda : E \rightarrow \mathbb{R}$ defined by

$$\Phi_\lambda(u) := A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

The following two variant fountain theorems were established in [27].

Theorem 2.4 ([27, Theorem 2.2]). *Assume that the functional Φ_λ defined above satisfies*

(T₁) Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$,

(T₂) $B(u) \geq 0$ for all $u \in E$, and $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of E ,

(T₃) There exist $\rho_k > r_k > 0$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1, 2]$$

and

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ uniformly for } \lambda \in [1, 2].$$

Then there exist $\lambda_n \rightarrow 1$, $u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \rightarrow \eta_k \in [\xi_k(2), \beta_k(1)] \quad \text{as } n \rightarrow \infty.$$

Particularly, if $\{u_{\lambda_n}\}$ has a convergent subsequence for every k , then Φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\Phi_1(u_k) \rightarrow 0^-$ as $k \rightarrow \infty$.

Theorem 2.5 ([27, Theorem 2.1]). Assume that the functional Φ_λ defined above satisfies

(F₁) Φ_λ maps bounded sets to bounded sets for $\lambda \in [1, 2]$, and $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$,

(F₂) $B(u) \geq 0$ for all $u \in E$, Moreover, $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$,

(F₃) There exist $r_k > \rho_k > 0$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_\lambda(u), \quad \forall \lambda \in [1, 2].$$

Then

$$\alpha_k(\lambda) \leq \zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2],$$

where $B_k = \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) \mid \gamma \text{ is odd, } \gamma|_{\partial B_k} = id\}$.

Moreover, for a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \rightarrow 0 \text{ and } \Phi_\lambda(u_m^k(\lambda)) \rightarrow \zeta_k(\lambda) \text{ as } m \rightarrow \infty.$$

In order to apply the above two theorems to prove our main results, we define the functionals A , B and Φ_λ on our working space E by

$$A(u) = \frac{1}{2}\|u^+\|^2, \quad B(u) = \frac{1}{2}\|u^-\|^2 + \int_0^T W(t, u)dt, \quad (2.12)$$

and

$$\Phi_\lambda(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u^+\|^2 - \lambda \left(\frac{1}{2}\|u^-\|^2 + \int_0^T W(t, u)dt \right) \quad (2.13)$$

for all $u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+$ and $\lambda \in [1, 2]$. From Proposition 2.3, we know that $\Phi_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$. Let $X_j = \text{span}\{e_j\}$ for all $j \in \mathbb{N}$, where $\{e_n : n \in \mathbb{N}\}$ is the system of eigenfunctions given below (2.1). Note that $\Phi_1 = \Phi$, where Φ is the functional defined in (2.7).

2.2 Proof of Theorem 1.1

In this subsection, we will first establish the following lemmas and then give a proof of Theorem 1.1.

Lemma 2.6. *Let (AQ₁) and (AQ₃) be satisfied. Then $B(u) \geq 0$ for all $u \in E$ and $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace of E .*

Proof. Evidently, it follows from (2.12) and (AQ₁) that $B(u) \geq 0$ for all $u \in E$.

We claim that for any finite dimensional subspace $F \subset E$, there exists a constant $\epsilon > 0$ such that

$$m(\{t \in [0, T] : |u(t)| \geq \epsilon\|u\|\}) \geq \epsilon, \quad \forall u \in F \setminus \{0\}. \quad (2.14)$$

Here and in the sequel, $m(\cdot)$ always denotes the Lebesgue measure in \mathbb{R} . If not, for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$m(\{t \in [0, T] : |u_n(t)| \geq \|u_n\|/n\}) < 1/n.$$

Let $v_n = u_n/\|u_n\| \in F$ for all $n \in \mathbb{N}$. Then $\|v_n\| = 1$ for all $n \in \mathbb{N}$, and

$$m(\{t \in [0, T] : |v_n(t)| \geq 1/n\}) < 1/n, \quad \forall n \in \mathbb{N}. \quad (2.15)$$

Passing to a subsequence if necessary, we may assume $v_n \rightarrow v_0$ in E for some $v_0 \in F$ since F is of finite dimension. Evidently, $\|v_0\| = 1$. In view of Lemma 2.2 and the equivalence of any two norms on F , we have

$$\int_0^T |v_n - v_0| dt \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.16)$$

and

$$|v_0|_\infty > 0.$$

By the definition of norm $|\cdot|_\infty$, there exists a constant $\delta_0 > 0$ such that

$$m(\{t \in [0, T] : |v_0(t)| \geq \delta_0\}) \geq \delta_0. \quad (2.17)$$

For any $n \in \mathbb{N}$, let

$$\Lambda_n = \{t \in [0, T] : |v_n(t)| < 1/n\} \quad \text{and} \quad \Lambda_n^c = \mathbb{R} \setminus \Lambda_n = \{t \in [0, T] : |v_n(t)| \geq 1/n\}.$$

Set $\Lambda_0 = \{t \in [0, T] : |v_0(t)| \geq \delta_0\}$. Then for n large enough, by (2.15) and (2.17), we have

$$m(\Lambda_n \cap \Lambda_0) \geq m(\Lambda_0) - m(\Lambda_n^c) \geq \delta_0 - 1/n \geq \delta_0/2. \quad (2.18)$$

Consequently, for n large enough, there holds

$$\begin{aligned} \int_0^T |v_n - v_0| dt &\geq \int_{\Lambda_n \cap \Lambda_0} |v_n - v_0| dt \\ &\geq \int_{\Lambda_n \cap \Lambda_0} (|v_0| - |v_n|) dt \\ &\geq (\delta_0 - 1/n) \cdot m(\Lambda_n \cap \Lambda_0) \\ &\geq \delta_0^2/4 > 0. \end{aligned}$$

This is in contradiction to (2.16). Therefore (2.14) holds.

For the ϵ given in (2.14), let

$$\Lambda_u = \{t \in [0, T] : |u(t)| \geq \epsilon \|u\|\}, \quad \forall u \in F \setminus \{0\}.$$

Then by (2.14),

$$m(\Lambda_u) \geq \epsilon, \quad \forall u \in F \setminus \{0\}. \quad (2.19)$$

By (AQ₃), there exists a constant $R_3 > R_1$ such that

$$W(t, u) \geq d|u|/2, \quad \forall t \in [0, T] \text{ and } |u| \geq R_3, \quad (2.20)$$

where R_1 is the constant given in (AQ₁). Note that

$$|u(t)| \geq R_3, \quad \forall t \in \Lambda_u \quad (2.21)$$

for any $u \in F$ with $\|u\| \geq R_3/\epsilon$. Combining (AQ₁), (2.19) and (2.21), for any $u \in F$ with $\|u\| \geq R_3/\epsilon$, we have

$$\begin{aligned} B(u) &= \frac{1}{2} \|u^-\|^2 + \int_0^T W(t, u) dt \\ &\geq \int_{\Lambda_u} W(t, u) dt \\ &\geq \int_{\Lambda_u} d|u|/2 dt \\ &\geq d\epsilon \|u\| \cdot m(\Lambda_u)/2 \geq d\epsilon^2 \|u\|/2. \end{aligned}$$

This implies $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$ on any finite dimensional subspace $F \subset E$. The proof is complete. \square

Lemma 2.7. *Assume that (AQ₁)–(AQ₃) hold. Then there exists a positive integer k_1 and two sequences $0 < r_k < \rho_k \rightarrow 0$ as $k \rightarrow \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_1, \quad (2.22)$$

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ uniformly for } \lambda \in [1, 2] \quad (2.23)$$

and

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \quad (2.24)$$

where $Y_k = \oplus_{j=1}^k X_j = \text{span}\{e_1, \dots, e_k\}$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j} = \overline{\text{span}\{e_k, \dots\}}$ for all $k \in \mathbb{N}$.

Proof. We complete the proof via the following two steps.

Step 1. We prove (2.22) and (2.23).

Note first that $Z_k \subset E^+$ for all $k \geq \bar{n} + 1$ by (2.5), where \bar{n} is the integer defined in (2.2). By (2.6), for any $u \in E$ with $\|u\| \leq R_2/\tau_\infty$, there holds

$$\|u\|_\infty \leq R_2, \quad (2.25)$$

where R_2 and τ_∞ are the constants in (AQ₂) and (2.6) respectively. Then for any $k \geq \bar{n} + 1$ and $u \in E^+$ with $\|u\| \leq R_2/\tau_\infty$, by (AQ₂) and (2.13), we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2 \int_0^T W(t, u) dt \\ &\geq \frac{1}{2}\|u\|^2 - 2c_2|u|_1, \quad \forall \lambda \in [1, 2]. \end{aligned} \quad (2.26)$$

Let

$$\ell_k = \sup_{u \in Z_k, \|u\|=1} |u|_1, \quad \forall k \in \mathbb{N}. \quad (2.27)$$

Then

$$\ell_k \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (2.28)$$

since E is compactly embedded into L^1 . Consequently, (2.26) and (2.27) imply

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2c_2\ell_k\|u\| \quad (2.29)$$

for all $k \geq \bar{n} + 1$ and $u \in E^+$ with $\|u\| \leq R_2/\tau_\infty$. For any $k \in \mathbb{N}$, let

$$\rho_k = 8c_2\ell_k. \quad (2.30)$$

Then by (2.28), we have

$$\rho_k \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (2.31)$$

Evidently, there exists a positive integer $k_1 > \bar{n} + 1$ such that

$$\rho_k < R_2/\tau_\infty, \quad \forall k \geq k_1. \quad (2.32)$$

For any $k \geq k_1$, (2.29) together with (2.30) and (2.32) yields

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0.$$

By (2.29), for any $k \geq k_1$ and $u \in Z_k$ with $\|u\| \leq \rho_k$, we have

$$\Phi_\lambda(u) \geq -2c_2\ell_k\rho_k.$$

Observing that $\Phi_\lambda(0) = 0$ by (AQ₂), then

$$0 \geq \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \geq -2c_2\ell_k\rho_k, \quad \forall k \geq k_1.$$

This together with (2.28) and (2.31) implies

$$\xi_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ uniformly for } \lambda \in [1, 2].$$

Step 2. We show that (2.24) holds.

For any $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that

$$|u|_2 \geq C_k\|u\|, \quad \forall u \in Y_k \quad (2.33)$$

since norms $|\cdot|$ and $\|\cdot\|$ are equivalent on finite dimensional space Y_k . By (AQ₂), for any $k \in \mathbb{N}$, there exists a constant $\delta_k > 0$ such that

$$W(t, u) \geq |u|^2/C_k^2, \quad \forall |u| \leq \delta_k. \quad (2.34)$$

By (2.6), for any $k \in \mathbb{N}$ and $u \in E$ with $\|u\| \leq \delta_k/\tau_\infty$, it holds

$$|u|_\infty \leq \delta_k,$$

where τ_∞ is the constant in (2.6). Combining this with (2.13), (2.33) and (2.34), for any $k \in \mathbb{N}$ and $u \in Y_k$ with $\|u\| \leq \delta_k/\tau_\infty$, we have

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_0^T W(t, u)dt \\ &\leq \frac{1}{2}\|u\|^2 - |u|_2^2/C_k^2 \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2, \quad \forall \lambda \in [1, 2]. \end{aligned} \quad (2.35)$$

Now for any $k \in \mathbb{N}$, if we choose

$$0 < r_k < \min \{ \rho_k, \delta_k / \tau_\infty \},$$

then (2.35) implies

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) \leq -r_k^2/2 < 0, \quad \forall k \in \mathbb{N}.$$

The proof is complete. \square

Proof of Theorem 1.1. In view of (2.8), (2.13) and Lemma 2.2, Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Evidently, $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$ since $W(t, u)$ is even in u . Thus the condition (T_1) of Theorem 2.4 holds. Lemma 2.6 shows that the condition (T_2) holds, while Lemma 2.7 implies that the condition (T_3) holds for all $k \geq k_1$, where k_1 is given there. Therefore, by Theorem 2.4, for each $k \geq k_1$, there exist $\lambda_n \rightarrow 1$, $u_{\lambda_n} \in Y_n$ such that

$$\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \rightarrow \eta_k \in [\xi_k(2), \beta_k(1)] \quad \text{as } n \rightarrow \infty. \quad (2.36)$$

For the sake of notational simplicity, throughout the remaining proof of Theorem 1.1 we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$.

Claim 1. $\{u_n\}$ is bounded in E .

Indeed, for the constant R_3 given in (2.20), there exists a constant $M_1 > 0$ such that

$$\left| W(t, u) - \frac{1}{2} \langle \nabla_u W(t, u), u \rangle \right| \leq M_1, \quad \forall t \in [0, T] \text{ and } |u| \leq R_3. \quad (2.37)$$

By virtue of (2.11), (2.13), (2.36), (2.37) and (AQ_1) , we have

$$\begin{aligned} -\Phi_{\lambda_n}(u_n) &= \frac{1}{2} \Phi'_{\lambda_n}|_{Y_n}(u_n) u_n - \Phi_{\lambda_n}(u_n) \\ &= \lambda_n \int_0^T \left[W(t, u_n) - \frac{1}{2} \langle \nabla_u W(t, u_n), u_n \rangle \right] dt \\ &\geq \lambda_n \int_{\Omega_n} \left[W(t, u_n) - \frac{1}{2} \langle \nabla_u W(t, u_n), u_n \rangle \right] dt - \lambda_n M_1 T \\ &\geq \frac{\lambda_n(2-\mu)}{2} \int_{\Omega_n} W(t, u_n) dt - \lambda_n M_1 T \\ &\geq \frac{d\lambda_n(2-\mu)}{4} \int_{\Omega_n} |u_n| dt - \lambda_n M_1 T, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\Omega_n := \{t \in [0, T] : |u_n(t)| \geq R_3\}$, and d, R_3 are the constants in (2.20). Combining this with (2.36), there exists a constant $M_2 > 0$ such that

$$\int_{\Omega_n} |u_n| dt \leq M_2, \quad \forall n \in \mathbb{N}. \quad (2.38)$$

For any $n \in \mathbb{N}$, let $\chi_n : [0, T] \rightarrow \mathbb{R}$ be the indicator function of Ω_n , that is,

$$\chi_n(t) = \begin{cases} 1, & t \in \Omega_n, \\ 0, & t \notin \Omega_n, \end{cases}.$$

Then by the definition of Ω_n and (2.38), there hold

$$|(1 - \chi_n)u_n|_\infty \leq R_3 \quad \text{and} \quad |\chi_n u_n|_1 \leq M_2, \quad \forall n \in \mathbb{N}. \quad (2.39)$$

By virtue of Remark 2.1, Lemma 2.2 and the Hölder inequality, we have

$$\begin{aligned} |u_n^- + u_n^0|_2^2 &= (u_n^- + u_n^0, u_n)_2 \\ &= (u_n^- + u_n^0, (1 - \chi_n)u_n)_2 + (u_n^- + u_n^0, \chi_n u_n)_2 \\ &\leq |(1 - \chi_n)u_n|_\infty |u_n^- + u_n^0|_1 + |\chi_n u_n|_1 |u_n^- + u_n^0|_\infty \\ &\leq c_3(R_3 + M_2) |u_n^- + u_n^0|_2, \quad \forall n \in \mathbb{N} \end{aligned}$$

for some $c_3 > 0$, where the last inequality follows from (2.39) and the equivalence of any two norms on finite dimensional space $E^- \oplus E^0$. Consequently, we get

$$|u_n^- + u_n^0|_2 \leq c_3(R_3 + M_2), \quad \forall n \in \mathbb{N}.$$

In view of the equivalence of norms $\|\cdot\|$ and $|\cdot|_2$ on $E^- \oplus E^0$ again, there exists a constant $M_3 > 0$ such that

$$\|u_n^- + u_n^0\| \leq M_3, \quad \forall n \in \mathbb{N}. \quad (2.40)$$

Note that

$$\|u_n^+\|^2 = 2\Phi_{\lambda_n}(u_n) + \lambda_n \|u_n^-\|^2 + 2\lambda_n \int_0^T W(t, u_n) dt, \quad \forall n \in \mathbb{N}.$$

Thus by (2.6), (2.8), (2.36) and (2.40), there holds

$$\begin{aligned} \|u_n\|^2 &= \|u_n^+\|^2 + \|u_n^- + u_n^0\|^2 \\ &= 2\Phi_{\lambda_n}(u_n) + \lambda_n \|u_n^-\|^2 + \|u_n^- + u_n^0\|^2 + 2\lambda_n \int_0^T W(t, u_n) dt \\ &\leq M_4 + 4c_1 |u_n|_\mu^\mu \\ &\leq M_4 + 4c_1 \tau_\mu^\mu \|u_n\|^\mu, \quad \forall n \in \mathbb{N} \end{aligned} \quad (2.41)$$

for some $M_4 > 0$, where τ_μ and c_1 are the constants in (2.6) and (2.8) respectively. Since $\mu < 2$, (2.41) yields that $\{u_n\}$ is bounded in E .

Claim 2. $\{u_n\}$ possesses a strong convergent subsequence in E .

In fact, by Claim 1, without loss of generality, we may assume

$$u_n^- \rightarrow u_0^-, \quad u_n^0 \rightarrow u_0^0, \quad u_n^+ \rightarrow u_0^+ \quad \text{and} \quad u_n \rightarrow u_0 \quad \text{as } n \rightarrow \infty \quad (2.42)$$

for some $u_0 = u_0^- + u_0^0 + u_0^+ \in E = E^- \oplus E^0 \oplus E^+$ since $\dim(E^- \oplus E^0) < \infty$. By virtue of the Riesz Representation Theorem, $\Phi'_{\lambda_n}|_{Y_n} : Y_n \rightarrow Y_n^*$ and $\Psi' : E \rightarrow E^*$ can be viewed as $\Phi'_{\lambda_n}|_{Y_n} : Y_n \rightarrow Y_n$ and $\Psi' : E \rightarrow E$ respectively, where Y_n^* is the dual space of Y_n . Note that

$$0 = \Phi'_{\lambda_n}|_{Y_n}(u_n) = u_n^+ - \lambda_n(u_n^- + P_n \Psi'(u_n)), \quad \forall n \in \mathbb{N},$$

where $P_n : E \rightarrow Y_n$ is the orthogonal projection for all $n \in \mathbb{N}$, that is,

$$u_n^+ = \lambda_n(u_n^- + P_n \Psi'(u_n)), \quad \forall n \in \mathbb{N}. \quad (2.43)$$

By Proposition 2.3, $\Psi' : E \rightarrow E$ is also compact. Due to the compactness of Ψ' and (2.42), the right-hand side of (2.43) converges strongly in E and hence $u_n^+ \rightarrow u_0^+$ in E . Combining this with (2.42), we have $u_n \rightarrow u_0$ in E . Thus Claim 2 is true.

Now from the last assertion of Theorem 2.4, we know that $\Phi = \Phi_1$ has infinitely many nontrivial critical points. Therefore, (1.1) possesses infinitely many nontrivial solutions by Proposition 2.3. The proof of Theorem 1.1 is complete. \square

2.3 Proof of Theorem 1.2

The following lemmas are needed in the proof of Theorem 1.2.

Lemma 2.8. *Let (SQ₁) and (SQ₂) be satisfied. Then $B(u) \geq 0$ for all $u \in E$. Furthermore, $A(u) \rightarrow \infty$ or $B(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.*

Proof. Using the similar arguments of the proof of Lemma 2.6 with (AQ₁) and (AQ₃) replaced by (SQ₁) and (SQ₂), we can prove

$$B(u) \geq 0, \quad \forall u \in E$$

and

$$B(u) \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty$$

on any finite dimensional subspace of E . Consequently,

$$B(u) \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty \text{ on } E^- \oplus E^0$$

since $E^- \oplus E^0$ is of finite dimension. Combining this with (2.4) and (2.12), we have

$$A(u) \rightarrow \infty \text{ or } B(u) \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty.$$

The proof is completed. \square

Lemma 2.9. *Assume that (SQ₁)–(SQ₃) hold. Then there exists a positive integer k_2 and two sequences $r_k > \rho_k \rightarrow \infty$ as $k \rightarrow \infty$ such that*

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) > 0, \quad \forall k \geq k_2 \quad (2.44)$$

and

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) < 0, \quad \forall k \in \mathbb{N}, \quad (2.45)$$

where $Y_k = \oplus_{j=1}^k X_j = \text{span}\{e_1, \dots, e_k\}$ and $Z_k = \overline{\oplus_{j=k}^\infty X_j} = \overline{\text{span}\{e_k, \dots\}}$ for all $k \in \mathbb{N}$.

Proof. We divide the proof into two steps.

Step 1. We first prove (2.44).

By (2.9) and (2.13), we have

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{1}{2}\|u\|^2 - 2 \int_0^T W(t, u) dt \\ &\geq \frac{1}{2}\|u\|^2 - 2a_1(|u|_1 + |u|_\nu^\nu) - 2a_2T, \quad \forall (\lambda, u) \in [1, 2] \times E^+, \end{aligned} \quad (2.46)$$

where a_1, a_2 are the constants in (2.9). Let

$$\ell_\nu(k) = \sup_{u \in Z_k, \|u\|=1} |u|_\nu, \quad \forall k \in \mathbb{N}. \quad (2.47)$$

Then

$$\ell_\nu(k) \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (2.48)$$

since E is compactly embedded into L^ν . Note that

$$Z_k \subset E^+, \quad \forall k \geq \bar{n} + 1, \quad (2.49)$$

where \bar{n} is the integer given in (2.2). Combining (2.6), (2.46), (2.47) and (2.49), for $k \geq \bar{n} + 1$, we have

$$\Phi_\lambda(u) \geq \frac{1}{2}\|u\|^2 - 2a_1\tau_1\|u\| - 2a_2T - 2a_1\ell_\nu^\nu(k)\|u\|^\nu, \quad \forall (\lambda, u) \in [1, 2] \times Z_k, \quad (2.50)$$

where τ_1 is the constant given in (2.6). By (2.48), there exists a positive integer $k_2 \geq \bar{n} + 1$ such that

$$\rho_k := (16a_1\ell_\nu^\nu(k))^{1/(2-\nu)} > \max\{16a_1\tau_1 + 1, 16a_2T\}, \quad \forall k \geq k_2 \quad (2.51)$$

since $\nu < 2$. Evidently,

$$\rho_k \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (2.52)$$

Combining (2.50) and (2.51), direct computation shows

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\|=\rho_k} \Phi_\lambda(u) \geq \rho_k^2/4 > 0, \quad \forall k \geq k_2.$$

Step 2. We then verify (2.45).

Note that the proof of (2.14) does not involve the conditions (AQ₁) and (AQ₃). Therefore, it still holds here. Consequently, for any $k \in \mathbb{N}$, there exists a constant $\epsilon_k > 0$ such that

$$m(\Lambda_u^k) \geq \epsilon_k, \quad \forall u \in Y_k \setminus \{0\}, \quad (2.53)$$

where $\Lambda_u^k := \{t \in \mathbb{R} : |u(t)| \geq \epsilon_k\|u\|\}$ for all $k \in \mathbb{N}$ and $u \in Y_k \setminus \{0\}$. By (SQ₂), for any $k \in \mathbb{N}$, there exists a constant $S_k > 0$ such that

$$W(t, u) \geq |u|^2/\epsilon_k^3, \quad \forall |u| \geq S_k. \quad (2.54)$$

Combining (2.13), (2.53), (2.54) and (SQ₂), for any $k \in \mathbb{N}$ and $\lambda \in [1, 2]$, we have

$$\begin{aligned} \Phi_\lambda(u) &\leq \frac{1}{2}\|u^+\|^2 - \int_0^T W(t, u)dt \\ &\leq \frac{1}{2}\|u\|^2 - \int_{\Lambda_u^k} (|u|^2/\epsilon_k^3) dt \\ &\leq \frac{1}{2}\|u\|^2 - \epsilon_k^2\|u\|^2 m(\Lambda_u^k)/\epsilon_k^3 \\ &\leq \frac{1}{2}\|u\|^2 - \|u\|^2 = -\frac{1}{2}\|u\|^2 \end{aligned} \quad (2.55)$$

for all $u = u^- + u^0 + u^+ \in Y_k$ with $\|u\| \geq S_k/\epsilon_k$. Now for any $k \in \mathbb{N}$, if we choose

$$r_k > \max\{\rho_k, S_k/\epsilon_k\},$$

then (2.55) implies

$$\beta_k(\lambda) := \max_{u \in Y_k, \|u\|=r_k} \Phi_\lambda(u) \leq -r_k^2/2 < 0, \quad \forall k \in \mathbb{N}.$$

The proof is complete. \square

Proof of Theorem 1.2. It follows from (2.6), (2.9) and (2.13) that Φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. In view of the evenness of $W(t, u)$ in u , it holds that $\Phi_\lambda(-u) = \Phi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$. Thus the condition (F₁) of Theorem 2.5 holds. Besides, Lemma 2.8 and Lemma 2.9 show that the condition (F₂) and (F₃) hold respectively for all $k \geq k_2$, where k_2 is given in Lemma 2.9. Therefore, by Theorem 2.5, for any $k \geq k_2$ and a.e. $\lambda \in [1, 2]$, there exists a sequence $\{u_m^k(\lambda)\}_{m=1}^\infty \subset E$ such that

$$\sup_m \|u_m^k(\lambda)\| < \infty, \quad \Phi'_\lambda(u_m^k(\lambda)) \rightarrow 0 \text{ and } \Phi_\lambda(u_m^k(\lambda)) \rightarrow \zeta_k(\lambda) \quad \text{as } m \rightarrow \infty, \quad (2.56)$$

where

$$\zeta_k(\lambda) := \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} \Phi_\lambda(\gamma(u)), \quad \forall \lambda \in [1, 2]$$

with $B_k = \{u \in Y_k : \|u\| \leq r_k\}$ and $\Gamma_k := \{\gamma \in C(B_k, E) \mid \gamma \text{ is odd, } \gamma|_{\partial B_k} = id\}$. Furthermore, it follows from the proof of Lemma 2.9 that

$$\zeta_k(\lambda) \in [\bar{\alpha}_k, \bar{\zeta}_k], \quad \forall k \geq k_2, \quad (2.57)$$

where $\bar{\zeta}_k := \max_{u \in B_k} \Phi_1(u)$ and $\bar{\alpha}_k := \rho_k^2/4 \rightarrow \infty$ as $k \rightarrow \infty$ by (2.52). Using the similar arguments of the proof of Claim 2 in the proof of Theorem 1.1, for each $k \geq k_2$, we can choose $\lambda_n \rightarrow 1$ such that the sequence $\{u_m^k(\lambda_n)\}_{m=1}^\infty$ obtained by (2.56) has a strong convergent subsequence. Without loss of generality, we may assume

$$\lim_{m \rightarrow \infty} u_m^k(\lambda_n) = u_n^k \in E, \quad \forall n \in \mathbb{N} \text{ and } k \geq k_2.$$

This together with (2.56) and (2.57) yields

$$\Phi'_{\lambda_n}(u_n^k) = 0, \quad \Phi_{\lambda_n}(u_n^k) \in [\bar{\alpha}_k, \bar{\zeta}_k], \quad \forall n \in \mathbb{N} \text{ and } k \geq k_2. \quad (2.58)$$

As in the proof of Theorem 1.1, we claim that the sequence $\{u_n^k\}_{n=1}^\infty$ in (2.58) is bounded in E and possesses a strong convergent subsequence with the limit $u^k \in E$ for all $k \geq k_2$. In fact, by (SQ₃), there exist constants $L_0 > 0$ and $D_1 > 0$ such that

$$\frac{1}{2} \langle \nabla_u W(t, u), u \rangle - W(t, u) \geq \frac{b}{4} |u|^e, \quad \forall t \in [0, T] \text{ and } |u| \geq L_0 \quad (2.59)$$

and

$$\left| \frac{1}{2} \langle \nabla_u W(t, u), u \rangle - W(t, u) \right| \leq D_1, \quad \forall t \in [0, T] \text{ and } |u| \leq L_0. \quad (2.60)$$

For notational simplicity, we will set $u_n = u_n^k$ for all $n \in \mathbb{N}$ throughout this paragraph. By virtue of (2.11), (2.13) and (2.58)–(2.60), we have

$$\begin{aligned} \Phi_{\lambda_n}(u_n) &= \Phi_{\lambda_n}(u_n) - \frac{1}{2} \Phi'_{\lambda_n}(u_n) u_n \\ &= \lambda_n \int_0^T \left[\frac{1}{2} \langle \nabla_u W(t, u_n), u_n \rangle - W(t, u_n) \right] dt \\ &\geq \lambda_n \int_{\Pi_n} \left[\frac{1}{2} \langle \nabla_u W(t, u_n), u_n \rangle - W(t, u_n) \right] dt - \lambda_n D_1 T \\ &\geq \frac{b\lambda_n}{4} \int_{\Pi_n} |u_n|^\varrho dt - \lambda_n D_1 T, \quad \forall n \in \mathbb{N}, \end{aligned}$$

where $\Pi_n := \{t \in [0, T] : |u_n(t)| \geq L_0\}$. This together with (2.58) implies that

$$\int_{\Pi_n} |u_n|^\varrho dt \leq D_2, \quad \forall n \in \mathbb{N} \quad (2.61)$$

for some $D_2 > 0$. Since $\varrho \geq 1$, it also holds

$$\int_{\Pi_n} |u_n| dt \leq D_3, \quad \forall n \in \mathbb{N} \quad (2.62)$$

for some $D_3 > 0$ by using the Hölder inequality if necessary. Then the similar arguments of the proof of Claim 1 in the proof of Theorem 1.1 yields

$$\|u_n^- + u_n^0\| \leq D_4, \quad \forall n \in \mathbb{N} \quad (2.63)$$

for some $D_4 > 0$. Combining (2.6), (2.11), (2.61), (2.63) and (SQ₁), we have

$$\begin{aligned} \|u_n\|^2 &= \|u_n^+\|^2 + \|u_n^- + u_n^0\|^2 \\ &= \Phi'_{\lambda_n}(u_n) u_n + \lambda_n \|u_n^-\|^2 + \|u_n^- + u_n^0\|^2 + \lambda_n \int_0^T \langle \nabla_u W(t, u_n), u_n \rangle dt \\ &\leq D_5 + 2 \int_0^T a_1 (1 + |u_n|^{\nu-1}) |u_n| dt \\ &= D_5 + 2a_1 |u_n|_1 + 2a_1 \int_0^T |u_n|^\nu dt \\ &\leq D_5 + 2a_1 \tau_1 \|u_n\| + 2a_1 \left[\int_{[0, T] \setminus \Pi_n} |u_n|^\nu dt + \int_{\Pi_n} |u_n|^\nu dt \right] \\ &\leq D_5 + 2a_1 T L_0^\nu + 2a_1 \tau_1 \|u_n\| + 2a_1 |u_n|_\infty^{\nu-\varrho} \int_{\Pi_n} |u_n|^\varrho dt \\ &\leq D_5 + 2a_1 T L_0^\nu + 2a_1 \tau_1 \|u_n\| + 2a_1 D_2 \tau_\infty^{\nu-\varrho} \|u_n\|^{\nu-\varrho} \end{aligned} \quad (2.64)$$

for some $D_5 > 0$, where τ_1 and τ_∞ are the constants in (2.6). Since $\varrho > \nu - 2$, (2.64) implies that $\{u_n\}$ is bounded in E . By virtue of the similar arguments of the proof of Claim 2 in the proof of Theorem of 1.1 again, we see that $\{u_n\}$ has a strong convergent subsequence.

Now for each $k \geq k_2$, by (2.58), the limit u^k is just a critical point of $\Phi = \Phi_1$ with $\Phi(u^k) \in [\bar{\alpha}_k, \bar{\zeta}_k]$. Since $\bar{\alpha}_k \rightarrow \infty$ as $k \rightarrow \infty$ in (2.57), we get infinitely many nontrivial critical points of Φ . Therefore, (1.1) possesses infinitely many nontrivial solutions by Proposition 2.3. The proof of Theorem 1.1 is complete. \square

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